Flexible/inflexible: Clare and Mandy's story

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Research has shown that mental computation is a valid computational method which contributes to mathematical thinking as a whole (e.g., Sowder, 1990). This paper reports on a pilot study of young children's understanding of mental computation, and compares the mental architecture of two mental computers, one flexible and one inflexible. Further questions which have been raised as a result of this pilot study will also be discussed.

Recent research has suggested that mental computation should play a more prominent role in number strands of mathematics curricula (e.g., Cobb & Merkel, 1989; McIntosh, 1996; Reys & Barger, 1994; Sowder, 1990; Willis, 1990). Reasons for its inclusion are: mental computation enables children to learn how numbers work, make decisions about procedures, and create strategies (e.g., Reys, 1985; Sowder, 1990); mental computation promotes greater understanding of the structure of number and its properties (Reys, 1984); and it can be used as a "*vehicle* for promoting thinking, conjecturing, and generalizing based on conceptual understanding" (Reys & Barger, 1994, p. 31). In effect, mental computation promotes number sense (National Council of Teachers of Mathematics, 1989; Sowder, 1990). Further, mental computation has an utilitarian purpose (Clarke & Kelly, 1989). In fact, Willis (1992) suggested that mental computation should be the main form of computation, with written computation to serve as memory support.

Researchers (Kamii, Lewis, & Jones, 1991; Reys, Reys, Nohda, & Emori, 1995) have recommended that children should be free to formulate their own mental strategies, as understanding of algorithms is improved if children construct strategies in line with their own natural ways of thinking. McIntosh (1996) supported this and suggested that teaching mental strategies the same as formal pen and paper strategies have been taught in the past is not the solution to the present lack of attention given to mental computation.

Other researchers (Cooper, Heirdsfield, & Irons, 1996a, 1996b; Heirdsfield & Cooper, 1996) have reported the effects of pen and paper instruction on children's spontaneous mental strategies. Before instruction, children exhibited a variety of efficient strategies; whereas, after instruction, children tended to employ a mental strategy which appeared to reflect the pen and paper algorithm. The researchers also argued for a hierarchy of strategies, with *separation* strategies being least efficient, then *aggregation* and finally *wholistic* being the most efficient (Heirdsfield & Cooper, 1997). From this research, questions arise as to why some students are more accurate and flexible with more efficient strategies than others, and how their expertise relates to other number topics.

Connections have been drawn between mental computation and other factors, including number sense (McIntosh, 1996; McIntosh, Reys, & Reys, 1992; Reys, 1984; Sowder, 1990, 1992, 1994; Trafton, 1992), numeration and place value (McIntosh, 1996; Reys, 1985; Sowder, 1992), computational estimation (Heirdsfield, 1996; Reys, Bestgen, Rybolt, & Wyatt, 1982; Sowder & Wheeler, 1989), and number fact knowledge (Hope & Sherrill, 1987; Sowder & Wheeler, 1989). The results of year 4 children's mental computation, computational estimation, and number fact knowledge (Heirdsfield, 1996) indicated that children who were accurate and flexible in mental computation possessed advanced number fact skills (i.e., they were able to access basic facts using recall, or were able to employ advanced derived facts strategies). Further, these children were also proficient in computational estimation.

Hope (1987), Hope and Sherrill (1987), Reys (1985), and Sowder (1994) have identified characteristics of proficient mental computers. Skilled mental computers use a variety of strategies in different situations (depending on numbers and context), because they are disposed to making sense of mathematics (Sowder, 1994). Therefore, they must be aware of a variety of strategies and have the confidence to use them. There is also evidence of reflection and regulation. Hope (1987) and Dowker (1990) reported children and adults choosing strategies based on their knowledge of number and operations, and choosing appropriate strategies to deal with the problems.

The reasons that some children are unable to use better strategies than the pen and paper algorithms in different situations, vary. The study of good mental computers may go beyond cognition and metacognition, to affects and beliefs (Sowder, 1994).

In summary, research on mental computation and number has proposed connections among mental computation and number sense, particularly number facts, computational estimation, numeration, and properties of number and operation; social and affective issues including attributions, self efficacy, and social context (e.g., classroom and home); and metacognitive processes.

This paper reports on two children's responses in a research project designed to address issues such as: Why are some children better mental computers than others? What are some contributing factors?

Method

Subjects

Two children, Clare and Mandy, were selected from a population of 16 year 3 children from one classroom in an inner city Brisbane school, as a result of testing for accuracy and flexibility in mental computation. Clare was accurate and flexible, while Mandy was accurate, but inflexible, using a single strategy consistently.

Instruments

Mandy and Clare were given a series of indepth interviews. Tasks were given for numeration, operations, mental computation, number fact knowledge, computational estimation. Questions were asked self efficacy, beliefs, and metacognition. The children were also required to complete the Student Preference Survey (SPS) (McIntosh, 1996). In order to get a feel for classroom and home contexts, the children were encouraged to indulge in general conversation, and the teacher was invited to respond to initial and general inferences.

Procedure

Mandy and Clare were withdrawn from their classroom on a one to one basis, and interviewed in a quiet room. The interviews were videotaped, and each interview session lasted for no more than 30 minutes at a time. Because of the variety of aspects covered, each child received three interview sessions.

Analysis

Mental computation, computational estimation, and number fact responses were analysed for strategy choice, flexibility, accuracy, understanding of number and numeration, and metacognition. Number and operations tasks were analysed for understanding of associativity and inverses, and relationships (e.g., 69-43=26, ∴69-44=25). Analysis of students' responses to numeration tasks were based on Ross's five levels (1986). Analysis across individuals' interviews was undertaken to investigate connections with mental computation, for instance, whether understanding of noncanonical partitioning of numbers was used for mental computation.

Results

Clare's story

The strategies that Clare employed for mental computation revealed numeration understanding, knowledge of the effects of operations, and facility with number facts. These findings were supported by evidence from interviews which specifically addressed these aspects.

Clare appeared confident in computation, and stated she liked mathematics, because she found it easy and is therefore good at it, that is, she attributed her success to ability. When asked how she knew she was correct, Clare replied, "I just think I'm right. I am usually right." Thus, Clare attributed success to internal, stable factors. Clare possibly attributed failure (if any) to "very foolish mistakes", an unstable internal factor (lack of immediate effort) (Weiner, 1985).

Further, Clare *needed* to achieve, and only felt confident attempting questions if she believed she *could* succeed. After being unsuccessful at calculating 265-99 in the selection interview, she went home and asked her father how to calculate such examples. She was happy to attempt a similar question (234-99) in the next interview, because she now knew how to calculate it. However, she did not know why it worked ("That's what Dad told me to do."). Her confidence was also exhibited by her stating that her subtraction method (of levelling) "annoys Miss A...", but she was determined to continue to use it. However, she *did* realise that method was too complex for 3 digit examples. In the follow up mental computation interviews, when asked to think of another solution method, she saw no reason to think of a different method, except for the fun of it (appease the interviewer?). However, once she reasoned that some of her second methods were better than her first methods, she thought it was quite a good idea to indulge me. At times, hints had to be given, e.g., "what is 19 near?". Other times, no hints were given, for example, after solving 80-49 by 80-40= 40, take another 10, 10-9=1, 31, Clare then turned 49 into 50 and proceeded, 80-50+1. Clare's confidence in her ability and her reluctance (at first) to try a different method was reflected across all her classroom work. She had a strong preference for her own methods. Her later acceptance of alternative methods and even preference for these came as a shock to her teacher ("out of character for Clare"). It is suggested that she had nothing to prove to the interviewer by remaining adamant about the suitability or otherwise of her strategies. In discussions with her teacher, it was suggested that Clare was expected to succeed, but also to enjoy school and learning. Her parents did not consider it essential for Clare to be a high achiever, although she was.

Clare stated that she believed she would be able to solve the mental computation questions, and she could. This was reflected in her responses to the Student Preference Survey (SPS) (McIntosh, 1996). For all Clare's confidence, when asked to solve subtraction problems, she replied, "I don't particularly want to. I don't like doing take away in my head." This was despite the fact that she could. This attitude towards subtraction was reflected in her response in the SPS, where she responded positively to calculating mentally for only the simple subtraction problems.

Clare's ability to manipulate operations was not consistent. In the number and operations interviews, she was not always sure whether to add or subtract one when taking away one more or one less (e.g., 73-45=28, 74-46=?). Thus, although her father had shown her a method based on this principle, there was not solid understanding. The similar concept for addition (e.g., 234+99=333, because 234+100=334, and take 1, so 333), however, posed no problem for her.

Clare's number facts were fast and accurate. In the number facts test, she used recall and *derived facts strategies* (DFS), predominantly. One of the DFS, a levelling strategy (e.g., 17-9: take 7 out of 9 and out of 17, so 10-2=8), was similar to a subtraction strategy used in the mental computation interviews (e.g., 52-19: take 2 out 9 = 7, so 10-7=3, 40-10=30, 33). Clare stated she had not been taught this strategy, but had worked it out for herself. This offers support for children who employ DFS understand relationships between numbers, and are able to use this understanding of number properties in mental computation. Her agility with number facts was an advantage in the mental computation interviews, as working memory was available for efficiently solving more complex problems. Immediately before the indepth mental computation interviews, the children were presented with the number facts test, in which Clare calculated 15-8 by levelling. She was then able to recall this fact for the same question in the mental computation interview, that is, she had learnt from the experience.

The children's teacher was amazed that Clare had formulated the levelling strategy. She stated that she had used similar strategies when modelling addition tasks, but did not expect children to be able to use them either for addition or subtraction. It appeared that Clare had the capacity to build up a rich, interconnected network of knowledge, and access this knowledge, readily.

Clare defined estimation as a "type of guessing", a definition in common with other children in her class. She stated that she only estimated when given classroom estimation tasks that were treated as *rounding* only. However, Clare did not employ *rounding* in the interview. Rather, she used other strategies more appropriate to the situations, for instance, *truncation* and *wholistic*. Because Clare's mental computation was so good, she attempted to calculate accurately. This type of response has been reported elsewhere for proficient mental calculators (Heirdsfield, 1996; LeFevre, Greenham, & Waheed, 1993). It was decided to present Clare with additional 3 digit estimation questions that were too difficult for exact calculation. Clare's responses reflected an understanding of magnitude of number, place value, and the effect of operations. One example of a successfully completed task was: "Your friend has $152 and spends $144 on a cassette recorder. You have $156 and spend $142 on another cassette recorder. Who has more money left?" Response: "I do, because I started with more and spent less."

The numeration tasks revealed Clare's understanding of both canonical and noncanonical representations of number (Ross, 1986). She was particularly flexible with different representations of such numbers as 560 (5x100 + 6x10 + 0x1; 56x10 + 0x1; 500x1 + 6x10; 55x10 + 10x1; 5x100 + 3x10 + 30x1) and 209 (2x100 + 0x10 + 9x1; 20x10 + 9x1; 209x1; 19x10 + 19x1). Although MAB (Multibase Arithmetic Blocks) were available, Clare did not use them. However, there were times the interviewer had to encourage her to elicit more combinations, and she appeared to delight in the challenge.

Sowder (1994) suggested "(f)lexibility is but a manifestation of self-monitoring". However, Clare admitted that she generally employed the first method "that pops into my head"; therefore, there were times she chose an arguably less efficient mental strategy. Nevertheless, later in the interviews, such statements as, "why didn't I think of that in the first place?" indicated she began to consider strategy choice more carefully. During the course of interviews, Clare verbalised her thoughts, for instance, "No, that can't be right", "I'm lost now", "I'm usually right", "This one's difficult", "This one's easier", "I like this one, because it has something to do with 99", and "Seventy-five is easier to use than 76, so I'll use 75". These statements revealed the existence of metacognitive processes and beliefs. Clare had access to a variety of strategies, but stated that she rarely consciously chose the most appropriate strategy for the number context. However, when encouraged to think of other strategies, she made judgements regarding the suitability of the strategies. Clare was confident in experimenting with different strategies. She seemed to disregard what was taught in the classroom, rarely using the taught algorithm to solve the problems mentally. In fact, Clare revealed that she often used her levelling strategy for subtraction to solve written exercises.

Mandy's story

The other child, reported here, was Mandy who was also accurate in mental computation, but consistently employed a mental image of the pen and paper algorithm (i.e., she imagined the numbers one under the other, as if using pen and paper. The individual numbers were first separated into place values and then operated on by moving right to left). Mandy attributed her success to effort and practice. To check her work, she said she would check her answers by working through the examples the same way, and then wait for feedback from the teacher. Thus, she attributed success to internal, stable factors; however, although Mandy appeared to attribute failure also to an internal factor, she relied on teacher feedback (external factor) (Weiner, 1985).

Mandy stated she would be able to complete the tasks mentally, and could, although she was not particularly confident with subtraction. On the other hand, Mandy had to be deliberately encouraged to use strategies other than "calculating operations" (the term she used for pen and paper strategies). Mandy was successful at completing such tasks as: 257-100=157, so what does 257-99=? (after a good deal of thought), but she stated that she still preferred "using operations". The similar concept for addition (e.g., 234+99=333, because 234+100=334, and take 1, so 333) posed no problem for her. However, Mandy could not and would not apply the concept for the mental computation tasks. In discussions with Mandy's teacher, it was revealed that Mandy had high expectations for accuracy and speed when completing tasks. This could explain her using the same "automatic" procedure for solutions, and maintaining confidence in this procedure. Also, expectations from home could be summarised as "succeed at any cost". Mandy was expected to work hard, do as the teacher told her, and succeed (i.e., score well in tests).

Mandy had employed a mental image of the pen and paper algorithm in the selection interviews, and stated several times that she preferred that method and found it easier, as she was "used to it". Through prompting, Mandy developed a left to right aggregation strategy (e.g., 63-29: 63-20=53, 43; 43-9=34), and started to use it later in the interviews, because she said she wanted to practise the new way which may be easier for mental calculations; although she stated she still preferred the "old way". In fact, when employing alternative strategies, she still imagined the numbers one under the other, as though setting the examples out on paper.

Mandy used counting and some recall in both the number facts test and the mental computation interviews. She was also slower than Clare. This was despite Mandy's expectation for speed and accuracy. However, she did not have problems with memory overload when computing mentally. It is interesting to note that, in the mental computation interview, she recalculated the answers to the number facts (using similar strategies to those employed in the number facts test), although she had already done so, not 5 minutes before. She had not remembered the number facts solutions, nor had she made any links between what she had done previously and the task at hand.

Mandy was generally unsuccessful at the estimation tasks, and she could only relate estimation to measurement.

In the numeration tasks, Mandy was slow at representing numbers in different ways. She had to be prompted with such questions as, "What about some ones?", and needed the support of MAB for many examples. Even with MAB, she did not show a solid understanding of grouping and regrouping, as she constantly checked and recounted her manipulations. An alternative explanation for her constantly recounting blocks could be that her need for absolute certainty overshadowed her understanding of number. However, it appears curious that she would have to count and recount tens to ones, if she truly understood regrouping.

Concluding comments and further questions

Although both children possessed a high degree of accuracy in mental computation, Clare who exhibited flexibility contrasted in many ways to Mandy who employed a mental image of the pen and paper algorithm. Some of these differences are now summarised (Table 1) and further issues discussed.

Table 1

Comparison of Clare and Mandy

Aspect for comparison	Clare	Mandy
Accuracy	100% accuracy	100% accuracy
Flexibility	Used full hierarchy of strategies.	Used mental image of pen and paper algorithm.
Access to alternatives	Accessed valid and efficient strategies.	Had difficulty, but with scaffolding, developed an aggregation strategy.
Attribution	Attributed success to ability	Attributed success to effort.
Metacognition	Some evidence of conscious choices, reflection and evaluation.	Little evidence, except when conscious decision made to practise new mental strategy.
Self efficacy	Confident in her ability to solve tasks, and she could; although found subtraction more difficult. Confident in her own strategies.	Confident in her ability to solve tasks, and she could; although found subtraction more difficult. Confident in teacher taught strategies.
Numeration	Canonical and non canonical understanding, without concrete aids. Flexible with different representations.	Poor understanding, relied on concrete representations.
Number and operation	Not consistent, better understanding with addition.	Showed some understanding in number and operation interview, but did not apply in mental computation tasks.
Computational estimation	Proficient. Exhibited good number sense.	Poor.
Number facts	Accuracy and speed evident. Employed DFS and recall.	Accuracy, but less speed. Employed recall and count.

It is interesting to note that the aspects of number and operation, and numeration which were investigated in relation to mental computation, are not taught in school. Mandy would not have these understandings, because she would not have been directed to do so by the teacher. On the other hand, Clare had access to a variety of strategies which required various understanding. What she had been taught in the classroom did not prevent her from exploring other possibilities. This was also evident in the numeration interview, where Clare exhibited understanding, beyond what would be presented in a year

3 classroom. It is theorised that a more complete understanding of numeration is necessary for accurate and flexible mental computation, but little or no understanding may be necessary for the use of the pen and paper algorithm.

Clare appeared to have a well connected network of knowledge of number, and was able to apply this knowledge across the different tasks, for example, number facts were used in the mental computation interviews, numeration understanding (multiplicative nature of number system, noncanonical number representations) was applied in mental computation, computational estimation indicated a feel for number which was also reflected in mental computation, number and operation (although not well advanced) was at times reflected in mental computation. None of this evident in Mandy.

Because Clare possessed well structured and connected knowledge, her long term memory was more easily accessed for strategic knowledge, resulting in less load on working memory (the mechanism responsible for both temporary storage and processing of information). Mental computation requires concurrent processing and storage of information, that is, it is cognitively demanding. Mandy avoided this cognitive demand problem by relying on an automatic strategy. Further, she appeared to have well developed temporary storage. Mandy could remember partial sums and differences, even though she worked consistently right to left. However, she admitted this method was taxing on her memory. Clare appeared to be superior in processing of information. She may not have made conscious decisions regarding strategies, but there is evidence of something more than habitual patterns. The model hypothesised by Baddeley (1986) and Logie (1995) suggested that the central executive plays a role in reasoning, and is involved in the allocation of attention and automatic retrieval. It is attention, though that is involved in higher level cognitive tasks. The central executive interplays with the knowledge base, and it is this interplay that results in learning and acquisition of new knowledge. This certainly was evident with Clare. She attended to the task, accessed her knowledge base for strategies (and at times facts), processed the information, and much of this information was then stored in long term memory for future use, for instance, number fact recall or strategy use, when applicable. In fact, Clare would have overridden any habitual response (e.g., pen and paper algorithm) when it was necessary to employ more efficient strategies.

References

Baddeley, A. (1986). *Working memory.* New York: Oxford University Press.

Clarke, O., & Kelly, B. (1989). Calculators in the primary school - Time has come. In B. Doig (Ed.), *Everyone counts.* Parkville: Mathematics Association of Victoria.

Cobb, P., & Merkel, G. (1989). Thinking strategies: Teaching arithmetic through problem solving. In P. Trafton & A. Schulte (Eds.), *New directions for elementary school mathematics. 1989 yearbook.* Reston: National Council of Teachers of Mathematics.

Cooper, T. J., Heirdsfield, A.M., & Irons, C. J (1996a). Children's mental strategies for addition and subtraction word problems. In J. Mulligan & M. Mitchelmore (Eds.), *Children's number learning.* (pp. 147-162). Adelaide: Australian Association of Mathematics Teachers, Inc.

Cooper, T. J., Heirdsfield, A. M., & Irons, C. J. (1996b). Years 2 and 3 children's correct-response mental strategies for addition and subtraction word problems and algorithmic exercises. In L. Puig & A. Guiterrez (Eds.), *International Group for the Psychology of Mathematics Education, 20*(2), 241-248). Valencia: University of Valencia.

Dowker, A. (1990). *The variability of mathematicians' estimation strategies: Some cognitive implications.* Unpublished manuscript.

Heirdsfield, A. M. (1996). *Mental computation, computational estimation, and number fact knowledge for addition and subtraction in year 4 children.* Unpublished master's thesis, Queensland University of Technology, Brisbane.

Heirdsfield, A.M., & Cooper, T. J. (1996). The 'ups' and 'downs' of subtraction: Young children's additive and subtractive mental strategies for solutions of subtraction word problems and algorithmic exercises. *Annual Conference of the Mathematics Education Research Group of Australasia, 19*, 261-268. Melbourne: Deakin University Press.

Heirdsfield, A.M., & Cooper, T. J. (December, 1997). The architecture of mental addition and subtraction. *Paper presented at the Annual Meeting of the Australian Association for Research in Education.* Brisbane.

Hope, J. A. (1987). A case of a highly skilled mental calculator. *Journal for Research in Mathematics Education, 18*(5), 331-342.

Hope, J. A., & Sherrill, J. M. (1987). Characteristics of unskilled and skilled mental calculators. *Journal for Research in Mathematics Education, 18*(2), 98-111.

Kamii, C., Lewis, B., & Jones, S. (1991). Reform in primary education: A constructivist view. *Educational Horizons. 70*(1), 19-26.

LeFevre, J., Greenham, S. L., & Waheed, N. (1993). The development of procedural and conceptual knowledge in computational estimation. *Cognition and Instruction, 11*(2), 95-132.

Logie, R. H. (1995). *Visuo-spatial working memory.* Hove, UK: Erlbaum.

McIntosh, A. (1996). Mental computation and number sense of Western Australian students. In J. Mulligan & M. Mitchelmore (Eds.), *Children's number learning.* (pp. 259-276). Adelaide: Australian Association of Mathematics Teachers, Inc.

McIntosh, A., Reys, B., & Reys, R. (1992). A proposed framework for examining basic number sense. *For the Learning of Mathematics, 12*, 2-8.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: NCTM.

Reys, B. J. (1985). Mental computation. *Arithmetic Teacher, 32*(6), 43-46.

Reys, B. J., & Barger, R. H. (1994). Mental computation: Issues from the United States perspective. In R. E. Reys & N. Nohda (Eds.), *Computational alternatives for the twenty-first century.* Reston, Virginia: The National Council of Teachers of Mathematics.

Reys, R. E. (1984). Mental computation and estimation: past, present and future. *Elementary School Journal, 84*(5), 546-557.

Reys, R. E., Bestgen, B. J., Rybolt, J. F., & Wyatt, J. W. (1982). Processes used by good computational estimators. *Journal for Research in Mathematics Education, 13*(3), 183-201.

Reys, R. E., Reys, B. J., Nohda, N., & Emori, H. (1995). Mental computation performance and strategy use of Japanese students in grades 2, 4, 6, and 8. *Journal for Research in Mathematics Education, 26*(4), 304-326.

Ross, S. H. (1986). *The development of children's place value numeration concepts in grades two through five.* Paper presented at the annual meeting of the American Educational Research Association. San Francisco, April.

Sowder, J. (1990). Mental computation and number sense. *Arithmetic Teacher, 37*(7), 18-20.

Sowder, J. (1992). Making sense of numbers in school mathematics. In G. Leinhardt, R. Putman, & R. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching.* Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Sowder, J. (1994). Cognitive and metacognitive processes in mental computation and computational estimation. In R. Reys & N. Nohda (Eds.), *Computational alternatives for the twenty-first century.* Reston, Virginia: NCTM.

Sowder, J., & Wheeler, M. (1989). The development of concepts and strategies used in computational estimation. *Journal for Research in Mathematics Education, 20*, 130-46.

Trafton, P. (1992). Using number sense to develop mental computation and computational estimation. In C. Irons (Ed.), *Challenging children to think when they compute.* Brisbane: Centre for Mathematics and Science Education.

Weiner, B. (1985). An attribution theory of achievement motivation and emotion. *Psychological Review, 92*(4), 548-73.

Willis, S. (1990). Numeracy and society: The shifting ground. In S. Willis (Ed.), *Being numerate: What counts?* Melbourne: ACER.

Willis, S. (1992). The national statement on mathematics for Australian schools: A perspective on computation. In C. Irons (Ed.), *Challenging children to think when they compute.* (pp. 1-13). Brisbane: Centre for Mathematics and Science Education, Queensland University of Technology.